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On the Rayleigh-Stokes Problem for Generalized Fractional Oldroyd-B Fluids

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Abstract. We consider the initial-boundary value problem for the velocity distribution of a unidirectional flow of a generalized Oldroyd-B fluid with fractional derivative model. It involves two different Riemann-Liouville fractional derivatives in time. The problem is studied in a general abstract setting, based on a reformulation as a Volterra integral equation with kernel represented in terms of Mittag-Leffler functions. Special attention is paid to the solution behavior in the scalar case, using some facts of the theory of the Bernstein functions. Numerical experiments are performed for different values of the parameters and plots are presented and discussed. The results are compared to those obtained in the limiting cases of generalized fractional Maxwell and second grade fluids.

INTRODUCTION

Various industrial and natural processes can be modelled as flows of non-Newtonian viscoelastic fluids: from polymer extrusion to processes in geophysics. The Oldroyd-B fluid model has become very popular since it can describe many of the non-Newtonian characteristics of polymers [1]. In recent years, the interest in the studies of viscoelastic materials with fractional derivative models has increased [2, 3, 4]. Fractional order models provide a higher level of adequacy, preserving linearity, and give the possibility for relatively simple description of the complex behaviour of non-Newtonian viscoelastic fluids. In [5] a very good fit with experimental data is achieved when the generalized fractional Oldroyd-B constitutive model is used.

The main subject of the present work will be the equation

$$(1 + aD_t^{\alpha})u_t = \mu(1 + bD_t^{\beta})\Delta u + f(x,t), \quad x \in \Omega, \ t > 0, \tag{1}$$

where D_t^{α} and D_t^{β} are Riemann-Liouville fractional time derivatives of orders $\alpha, \beta \in (0,1)$, $a, \mu > 0$, $b \geq 0$, $u_t = \partial u/\partial t$, Ω is a domain in $\mathbb{R}^d$, $d = 1, 2$, Δ is the Laplace operator acting on spatial variables. Here $u(x,t)$ is the velocity of a unidirectional flow and the given function $f(x,t)$ represents a driving force such as pressure gradient, electromagnetic field, etc. Equation (1) is obtained by substituting the constitutive equation for a generalized fractional Oldroyd-B fluid in the momentum equation for a unidirectional flow, see [6, 7, 8]. Initial and boundary conditions are also prescribed according to the assumed flow geometry.

Unidirectional flows of fractional Oldroyd-B fluids are studied in [6, 7, 8, 9, 10], where eigenfunction expansions of the solutions of equation (1) subject to different boundary conditions are obtained. In [10] estimates for the time-dependent components in these eigenfunction expansions are established. The particular case of equation (1) when $a = 0, b > 0$ is studied in [11] and [12], applying operator-theoretic approach. In [11] the Sobolev regularity of the solution is established for both smooth and nonsmooth initial data and in [12] the abstract version of this equation is investigated, based on the theory of completely monotone and Bernstein functions.

Let us note that the case $a = 0, b > 0$ corresponds to the generalized fractional second grade model, $b = 0, a > 0$ to the generalized fractional Maxwell model and $a = b = 0$ to Newtonian fluids. Regarding the values of the orders α and β of the fractional derivatives, different assumptions are given in the literature: in [5] the restriction $\alpha \geq \beta$ is

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imposed, while in other works, such as [8], the opposite restriction $\alpha \le \beta$ is assumed. Therefore we consider the whole range $\alpha, \beta \in (0, 1)$.

In the present work equation (1) is studied in a general abstract setting, based on a reformulation as a Volterra integral equation with kernel represented in terms of Mittag-Leffler functions.

The paper is organized as follows. After a section with preliminaries from Fractional Calculus, a derivation of equation (1) is presented. Next, the proper formulation of initial conditions for equation (1) is discussed, based on the asymptotic behaviour of the time-dependent components in the eigenfunction expansion. In the following section, the abstract Cauchy problem corresponding to equation (1) is considered and reformulated as an abstract Volterra integral equation. On the basis of some properties of its kernel, well-posedness is proven. The behaviour of the solution to the scalar equation is studied in the subsequent section, involving some facts from the theory of the Bernstein functions. To illustrate the theoretical results, numerical experiments are performed for different values of the parameters and plots are presented and discussed.

PRELIMINARIES

The sets of positive integers, real and complex numbers are denoted as usual by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, respectively, and $\mathbb{R}_+ = [0, \infty)$. Denote by Σ_θ the sector

$$\Sigma_\theta = \{s \in \mathbb{C};\ s \neq 0, |\arg s| < \theta\}.$$

Next some definitions and facts from Fractional Calculus are summarized, for details we refer to [13]. Let us first recall the definition of the fractional order Riemann-Liouville integral:

$$J_t^\gamma f(t) = \int_0^t \omega_\gamma(t-\tau) f(\tau)\, \mathrm{d}\tau, \quad \gamma > 0,$$

where

$$\omega_\gamma(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}, \quad \gamma > 0, t > 0. \tag{2}$$

Here $\Gamma(\cdot)$ is the Gamma function. The operators of fractional integration satisfy the semigroup property:

$$J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}, \quad J_t^\alpha \omega_\beta = \omega_{\alpha+\beta}, \quad \alpha, \beta > 0. \tag{3}$$

The Riemann-Liouville fractional derivative D_t^γ of order $\gamma \in (0, 1)$ is defined by

$$D_t^\gamma = D_t^1 J_t^{1-\gamma}, \quad \gamma \in (0, 1), \text{ where } \ D_t^1 = \mathrm{d}/\mathrm{d}t. \tag{4}$$

Recall the identities:

$$D_t^\gamma J_t^\gamma f(t) = f(t), \quad \gamma > 0; \quad J_t^\gamma D_t^\gamma f(t) = f(t) - (J_t^{1-\gamma} f)(0)\omega_\gamma(t), \quad \gamma \in (0, 1). \tag{5}$$

The application of the Laplace transform

$$\mathcal{L}\{f(t)\}(s) = \widehat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

to the operators of fractional order integration and differentiation gives

$$\mathcal{L}\{J_t^\gamma f\}(s) = s^{-\gamma} \widehat{f}(s), \quad \gamma > 0, \tag{6}$$

$$\mathcal{L}\{D_t^\gamma f\}(s) = s^{\gamma} \widehat{f}(s) - (J_t^{1-\gamma} f)(0), \quad \gamma \in (0, 1), \tag{7}$$

Denote as usual by $E_{\alpha,\beta}(\cdot)$ the Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta, z \in \mathbb{C}, \ \Re\alpha > 0. \tag{8}$$

The Mittag-Leffler function has the following asymptotic expansion for $\alpha \in (0, 2),\ \beta > 0$:

$$E_{\alpha,\beta}(-t) = -\sum_{k=1}^{N-1} \frac{(-t)^{-k}}{\Gamma(\beta - \alpha k)} + O(t^{-N}), \quad t \to +\infty, \tag{9}$$

Recall the Laplace transform pairs ($\Re\alpha > 0,\ \mu \in \mathbb{R},\ t > 0$)

$$\mathcal{L}\{\omega_\alpha(t)\}(s) = s^{-\alpha}, \tag{10}$$

and

$$\mathcal{L}\left\{t^{\beta-1} E_{\alpha,\beta}(-\mu t^\alpha)\right\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha + \mu}. \tag{11}$$

A function $f : (0, \infty) \to \mathbb{R}$ is called completely monotone if it is of class C^∞ and

$$(-1)^n f^{(n)}(t) \geq 0, \text{ for all } t > 0,\ n = 0, 1, ... \tag{12}$$

A C^∞ function $f : (0, \infty) \to \mathbb{R}$ is called a Bernstein function if it is nonnegative and its first derivative $f'(t)$ is a completely monotone function. The classes of completely monotone functions and Bernstein functions will be denoted by $\mathcal{CMF}$ and $\mathcal{BF}$, respectively. For details and references on these two classes of functions we refer to [14] and [15], Chapter 4.

The Mittag-Leffler function satisfies the following property for $t > 0$ (see, *e.g.,* [16])

$$E_{\alpha,\beta}(-t) \in \mathcal{CMF} \text{ iff } 0 \leq \alpha \leq 1,\ \alpha \leq \beta. \tag{13}$$

GOVERNING EQUATIONS AND DERIVATION DETAILS

For completeness, equation (1) is derived in this section. The 2D case is considered. For more details see also [6, 8].

The fundamental equations governing the unsteady motion of an incompressible viscoelastic fluid are the continuity equation:

$$\nabla \cdot \mathbf{V} = 0 \tag{14}$$

and the general Cauchy momentum equation:

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}\right) = \nabla \cdot \boldsymbol{\sigma}, \tag{15}$$

where $\mathbf{V}$ is the velocity vector, ρ is the fluid density and $\boldsymbol{\sigma}$ is the Cauchy stress tensor, which is given by

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau}. \tag{16}$$

Here p is the pressure and $\boldsymbol{\tau}$ is the shear stress tensor. In the case of a viscoelastic fluid with the generalized fractional Oldroyd-B model the following identity is satisfied

$$\left(1 + a\frac{D^\alpha}{Dt^\alpha}\right)\boldsymbol{\tau} = \eta\left(1 + b\frac{D^\beta}{Dt^\beta}\right)\mathbf{A_1}. \tag{17}$$

Here $\eta > 0$ is the dynamic viscosity of the fluid, $a = \lambda_1^\alpha,\ \ b = \lambda_2^\beta$ with λ_1 and λ_2 being the relaxation and retardation times, respectively, $\alpha, \beta \in (0, 1)$ are fractional parameters, and $\mathbf{A_1}$ is the first Rivlin-Ericksen tensor

$$\mathbf{A_1} = \nabla\mathbf{V} + (\nabla\mathbf{V})^T.$$

In (17), by $\dfrac{D^\gamma}{Dt^\gamma}$ the upper convected fractional time derivative is denoted, which is defined as:

$$\frac{D^\gamma \boldsymbol{\tau}}{Dt^\gamma} = D_t^\gamma \boldsymbol{\tau} + (\mathbf{V} \cdot \nabla)\boldsymbol{\tau} - (\nabla\mathbf{V}) \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot (\nabla\mathbf{V})^T. \tag{18}$$

Consider a unidirectional flow in the z - direction with velocity vector in the form $\mathbf{V} = (0, 0, u(x, y, t))$. Then the continuity equation (14) is satisfied automatically and the constitutive equation (17) gives:

$$(1 + aD_t^\alpha)\,\tau_{xz} = \eta\left(1 + bD_t^\beta\right)\frac{\partial u}{\partial x}, \quad (1 + aD_t^\alpha)\,\tau_{yz} = \eta\left(1 + bD_t^\beta\right)\frac{\partial u}{\partial y}. \tag{19}$$

Inserting (16) in the momentum equation (15) we obtain:

$$\rho\frac{\partial u}{\partial t} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} - \frac{\partial p}{\partial z}, \tag{20}$$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0. \tag{21}$$

Applying the operator $(1 + aD_t^\alpha)$ to both sides of (20) we deduce the equation

$$(1 + aD_t^\alpha)\rho\frac{\partial u}{\partial t} = (1 + aD_t^\alpha)\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}\right) - (1 + aD_t^\alpha)\frac{\partial p}{\partial z},$$

which, in view of the identities in (19) implies

$$(1 + aD_t^\alpha)\frac{\partial u}{\partial t} = \mu(1 + bD_t^\beta)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x, y, t), \tag{22}$$

where $\mu = \eta/\rho$ and

$$f(x, y, t) = -\frac{1}{\rho}(1 + aD_t^\alpha)\frac{\partial p}{\partial z},$$

i.e., the forcing function f here depends on a prescribed pressure gradient in the direction of the flow. In general, the function f can depend on other different factors inducing the flow. For example, in [8], where an electro-osmotic flow is considered, a similar equation is derived, with forcing function f represented in terms of an external electric field. The obtained equation (22) for the velocity distribution is the 2D variant of equation (1).

Depending on the assumed geometry, different boundary conditions can be prescribed. Here we restrict ourselves to rectangular domain $\Omega = (0, 1) \times (0, 1)$ and Dirichlet boundary conditions. This corresponds to flow in a duct of rectangular cross-section.

Regarding the initial conditions, initial velocity u at time $t = 0$ is prescribed, as in the case $a \neq 0$ one more initial condition is necessary. The second initial condition is usually assumed in the form: $u_t = 0$ at $t = 0$, see, *e.g.,* [6, 7, 8]. It is pointed out in [6] that this second initial condition has no physical significance and is adopted only for mathematical convenience. Whether this condition is appropriate for equation (1) is discussed further in the next section.

INITIAL CONDITIONS

Let Ω be a bounded rectangular domain in $\mathbb{R}^d$, $d = 1, 2$, with boundary $\partial\Omega$ and $x \in \Omega$ (this notation is adopted in the rest of the work, in contrast to the previous section, where by x the x axis was denoted). Let $T > 0$ be a fixed time, and $\alpha, \beta \in (0, 1)$, $a > 0$, $b \geq 0$, $\mu > 0$, be given constant parameters. Suppose $f \equiv 0$ and consider the equation

$$(1 + aD_t^\alpha)u_t = \mu(1 + bD_t^\beta)\Delta u, \quad x \in \Omega,\ t \in (0, T], \tag{23}$$

subject to homogeneous Dirichlet boundary condition

$$u(x, t) = 0, \quad x \in \partial\Omega,\ t \in [0, T], \tag{24}$$

and initial conditions

$$u(x, 0) = v(x), \quad x \in \overline{\Omega}, \tag{25}$$

$$u_t(x, 0) = 0, \quad x \in \overline{\Omega}. \tag{26}$$

Let $\{\lambda_n, \varphi_n(x)\}_{n\in\mathbb{N}}$ be the Dirichlet eigensystem of the operator $-\Delta$ on the domain Ω. Denote by $(.,.)$ the inner product in $L^2(\Omega)$. Applying eigenfunction decomposition, we are looking for solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\varphi_n(x), \tag{27}$$

where the functions $u_n(t)$ satisfy the following ordinary differential equation

$$(1 + aD_t^{\alpha})u_n'(t) = -\lambda_n\mu(1 + bD_t^{\beta})u_n(t), \tag{28}$$

$$u_n(0) = v_n, \;\; u_n'(0) = 0, \tag{29}$$

with $v_n = (v, \varphi_n)$. To solve this ODE we apply Laplace transform. Since $u_n(0) < \infty$ then $(J_t^{1-\beta}u_n)(0) = 0$ and identity (7) implies

$$\mathcal{L}\{D_t^{\beta}u_n\} = s^{\beta}\widehat{u_n}. \tag{30}$$

Analogously, since $u_n'(0) < \infty$ then

$$(J_t^{1-\alpha}u_n')(0) = 0 \tag{31}$$

and therefore

$$\mathcal{L}\{D_t^{\alpha}u_n'\} = s^{\alpha}\widehat{u_n'} = s^{\alpha}(s\widehat{u_n} - u_n(0)) = s^{\alpha}(s\widehat{u_n} - v_n). \tag{32}$$

Applying (30) and (32) we obtain from (28) the following equation for $\widehat{u_n}(s)$:

$$(1 + as^{\alpha})(s\widehat{u_n} - v_n) = -\lambda_n\mu(1 + bs^{\beta})\widehat{u_n}.$$

In this way we deduce

$$u_n(t) = v_nG_n(t), \;\; n \in \mathbb{N}, \tag{33}$$

where the function $G_n(t)$ is defined by its Laplace transform as follows:

$$\widehat{G_n}(s) = \frac{1 + as^{\alpha}}{s(1 + as^{\alpha}) + \mu\lambda_n(1 + bs^{\beta})}. \tag{34}$$

Let us check whether the initial conditions (29) are satisfied. From (34) and a property of the Laplace transform we deduce

$$G_n(0) = \lim_{s\to+\infty} s\widehat{G_n}(s) = 1,$$

thus the first initial condition is satisfied. Further,

$$\mathcal{L}\{G_n'\}(s) = s\widehat{G_n}(s) - G_n(0) = \frac{s(1 + as^{\alpha})}{s(1 + as^{\alpha}) + \mu\lambda_n(1 + bs^{\beta})} - 1 = \frac{-\mu\lambda_n(1 + bs^{\beta})}{s(1 + as^{\alpha}) + \mu\lambda_n(1 + bs^{\beta})}. \tag{35}$$

Therefore

$$\mathcal{L}\{G_n'\}(s) = O\left(s^{\beta-\alpha-1}\right), \;\; s \to +\infty,$$

and by applying Karamata-Feller Tauberian theorem ([17], Chapter XIII) it follows

$$G_n'(t) = O(t^{\alpha-\beta}), \;\; t \to 0.$$

This means that when $\alpha < \beta$ the function $G_n'(t)$ is singular at $t = 0$, *i.e.,* the second initial condition in (29) $u_n'(0) = 0$ is not satisfied. Thus, in general, initial condition of this type is not appropriate for the considered equation and should be replaced by another condition ensuring a unique solvability. We show now that the weaker condition (31) is a proper second initial condition for equation (28). First, we see from the above derivation that this condition is sufficient to obtain the solution u_n, given by (33)-(34). Second, the obtained function u_n always satisfies (31). Indeed, by the property of the Laplace transform (6) and (35)

$$\mathcal{L}\{J_t^{1-\alpha}G_n'\}(s) = s^{\alpha-1}\mathcal{L}\{G_n'\}(s) = \frac{-\mu\lambda_n(1 + bs^{\beta})s^{\alpha-1}}{s(1 + as^{\alpha}) + \mu\lambda_n(1 + bs^{\beta})} = O\left(s^{\beta-2}\right), \;\; s \to +\infty.$$

From here, applying the same Tauberian result as above we deduce

$$J_t^{1-\alpha} G_n'(t) = O(t^{1-\beta}), \quad t \to 0.$$

Since $\beta \in (0,1)$ then $(J_t^{1-\alpha} G_n')(0) = 0$ and thus (31) is satisfied for all $\alpha, \beta \in (0,1)$.

Therefore, in a mathematically rigorous formulation of an initial-boundary value problem for equation (23) the initial condition (26) should be replaced by a weaker condition. We have shown above that this can be a condition of the form

$$J_t^{1-\alpha} u_t = 0 \text{ at } t = 0, \; x \in \overline{\Omega}. \tag{36}$$

Due to nonlocality of the fractional integral in condition (36), it is less convenient for numerical implementations. Therefore, it should be noted that if the nonhomogeneous equation (1) is considered with sufficiently well-behaved function f and initial velocity $v = 0$, then the condition $u_t(x, 0) = 0$ still may be assumed and used for instance in finite-difference discretization of the problem.

Note that the eigenfunction expansion method used in this section works also with other than Dirichlet boundary conditions (*e.g.,* such as those considered in [8]). The time-dependent components $G_n(t)$ will retain the same form with λ_n being the corresponding eigenvalues.

ABSTRACT FORMULATION AS A VOLTERRA INTEGRAL EQUATION

To encompass different boundary conditions, corresponding to different flow geometries, in this section we consider the more general abstract form of the problem.

Let A be an unbounded closed linear operator densely defined on a complex Banach space X. Suppose that the operator A generates a one-parameter semigroup, which is bounded and analytic of angle $\pi/2$ (for the theory of one-parameter semigroups we refer to [18]). Note that the Laplace operator $A = \Delta$ with Dirichlet boundary conditions and considered in L^2- space setting, *i.e.,* with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, satisfies these conditions.

Consider the following abstract Cauchy problem

$$(1 + aD_t^\alpha) u_t = (1 + bD_t^\beta) Au, \; t > 0, \quad u(0) = v \in X, \;\; (J_t^{1-\alpha} u_t)(0) = 0. \tag{37}$$

It contains as a particular case problem (23)-(24)-(25)-(36) from the previous section and some other Rayleigh-Stokes problems for unidirectional flows of fractional Oldroyd-B fluids.

We show next that problem (37) is equivalent to the following abstract Volterra integral equation

$$u(t) = v + \int_0^t k(t - \tau) Au(\tau)\, d\tau \tag{38}$$

with scalar kernel $k(t)$ defined below in (43).

Applying the Laplace transform with respect to time, we obtain from (37) in a similar way as in the previous section the identity

$$(1 + as^\alpha)(s\widehat{u} - v) = (1 + bs^\beta) A\widehat{u}.$$

Therefore

$$\widehat{u}(s) = (1 + as^\alpha)\left(s(1 + as^\alpha) - (1 + bs^\beta)A\right)^{-1} v = \frac{g(s)}{s}\,(g(s) - A)^{-1}\, v, \tag{39}$$

where

$$g(s) = \frac{s(1 + as^\alpha)}{1 + bs^\beta}. \tag{40}$$

On the other hand, applying the Laplace transform to both sides of the integral equation (38) it follows

$$\widehat{u}(s) = \frac{1}{s}\left(1 - \widehat{k}A\right)^{-1} v. \tag{41}$$

Comparing (39) to (41) we obtain by the uniqueness of the Laplace transform that problem (37) is equivalent to the integral equation (38) with kernel $k(t)$, satisfying

$$\widehat{k}(s) = g(s)^{-1} = \frac{1 + bs^\beta}{s(1 + as^\alpha)}. \tag{42}$$

Taking the inverse Laplace transform of (42) we obtain by means of (11) the following explicit representation for the kernel $k(t)$ in terms of Mittag-Leffler functions:

$$k(t) = a^{-1} t^{\alpha} E_{\alpha,\alpha+1}(-a^{-1} t^{\alpha}) + a^{-1} b t^{\alpha-\beta} E_{\alpha,\alpha+1-\beta}(-a^{-1} t^{\alpha}). \tag{43}$$

Properties of the Kernel

From the definition of the Mittag-Leffler function (8) we obtain

$$k(t) = a^{-1} b \frac{t^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} + O\left(t^{\min\{\alpha, 2\alpha-\beta\}}\right), \quad t \to 0.$$

Therefore, $k \in L_{loc}(\mathbb{R}_+)$, as for $\alpha < \beta$ the kernel $k(t)$ has an integrable singularity at $t = 0$ and for $\alpha > \beta$ it is finite. Concerning large time behaviour, asymptotic expansion (9) implies

$$k(t) = 1 - a \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + b \frac{t^{-\beta}}{\Gamma(1-\beta)} + O\left(t^{-\min\{\alpha+\beta, 2\alpha\}}\right), \quad t \to +\infty.$$

In particular, the asymptotic expansions imply that

$$\int_0^{\infty} |k(t)| e^{-st} \, \mathrm{d}t < \infty \quad \text{for any} \quad s > 0.$$

The above properties of the kernel $k(t)$ are necessary for the application of the theory of abstract Volterra equations developed in [15] to equation (38).

Let us check whether the kernel has some properties related to complete monotonicity. Since from (13) $E_{\alpha,\alpha+1}(-t)$, $E_{\alpha,\alpha+1-\beta}(-t) \in \mathcal{CMF}$, the functions $E_{\alpha,\alpha+1}(-a^{-1}t^{\alpha})$ and $E_{\alpha,\alpha+1-\beta}(-a^{-1}t^{\alpha})$ are completely monotone for $t > 0$ (as compositions of completely monotone functions with the Bernstein function $a^{-1}t^{\alpha}$). Therefore, $k(t) \geq 0$. However, in general, we can not say more than that. As we can see on the plots presented on Figure 1 in general the kernel $k(t)$ is not a monotonic function.

To compute numerically and visualize the kernel $k(t)$ one possibility is to use its explicit representation (43) and some method for computation of the Mittag-Leffler functions (note that the series expansion (8) is not appropriate for numerical computation, especially for large times). Here we use another way of computation of the kernel $k(t)$ as a solution of an integral equation.

From (43) by the use of properties (3), (4), (8) (or from (42) by means of (7), (10), (11)) we deduce that the function $k(t)$ satisfies the equation

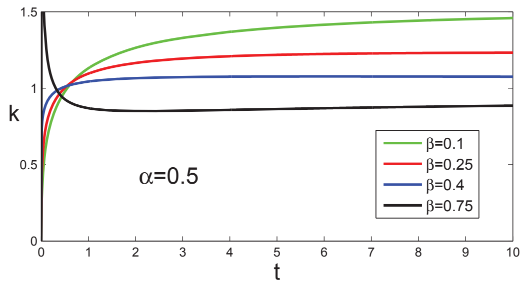
$$D_t^{\alpha} k(t) = -a^{-1} k(t) + a^{-1} \left(1 + b\omega_{1-\beta}(t)\right) \tag{44}$$

with initial condition $(J_t^{1-\alpha} k)(0) = 0$. Applying operator J_t^{α} to both sides of (44) and using the initial condition, semigroup property (3) and the second identity in (5), we obtain that the kernel $k(t)$ satisfies the following integral equation

$$k(t) = -a^{-1} \int_0^t \omega_{\alpha}(t-\tau) k(\tau) \, \mathrm{d}\tau + h(t), \quad \text{where} \quad h(t) = a^{-1} \left(\omega_{1+\alpha}(t) + b\omega_{1+\alpha-\beta}(t)\right). \tag{45}$$

This equation is used here for the numerical computation of the kernel $k(t)$, applying the fractional Adams method [19]: a predictor-corrector method in which as predictor the fractional Adams-Bashforth method is used and as corrector the fractional Adams-Moulton method.

On Figure 1 plots of the kernel $k(t)$ are presented for $a = b = 1$ and different values of the fractional parameters α and β. It is seen that the behaviour for $\alpha < \beta$ is qualitatively different from those for $\alpha > \beta$.



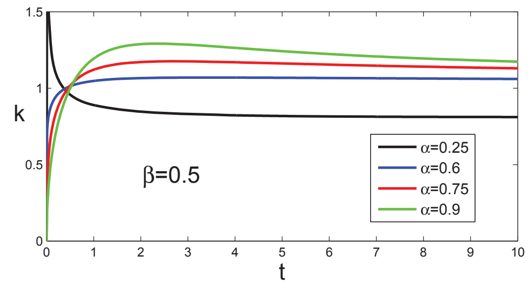


FIGURE 1. Plots of the kernel $k(t)$ for $a = b = 1$ and different values of α and β

Proof of Well-Posedness

Denote by $\|\cdot\|$ the norm in the Banach space X, and by $\rho(A)$ the resolvent set of the operator A. Further, we adopt the terminology from [15] about abstract Volterra integral equations.

We will prove that the Volterra integral equation (38) has a unique strong solution $u(t)$, which can be analytically extended to the sector Σ_{θ_0}, where

$$0 < \theta_0 < \frac{\pi(1-\alpha)}{2(1+\alpha)} < \frac{\pi}{2}, \tag{46}$$

and for any $\theta < \theta_0$ there exists a constant $C = C(\theta)$ such that $\|u(t)\| \le C\|v\|$ for all $t \in \Sigma_\theta$.

According to Theorem 2.1. in [15], for this it is sufficient to prove the following three properties:

(A1) $\widehat{k}(s)$ admits meromorphic extension to $\Sigma_{\theta_0+\pi/2}$;

(A2) $\widehat{k}(s) \neq 0$ and $1/\widehat{k}(s) \in \rho(A)$ for all $s \in \Sigma_{\theta_0+\pi/2}$;

(A3) For any $\theta < \theta_0$ there is a constant $C = C(\theta)$ such that the operator-valued function

$$H(s) = \frac{g(s)}{s}\left(g(s) - A\right)^{-1}, \quad \text{where} \quad g(s) = \frac{1}{\widehat{k}(s)}, \tag{47}$$

satisfies the estimate

$$\|H(s)\| \le C/|s| \quad \text{for all} \quad s \in \Sigma_{\theta+\pi/2}. \tag{48}$$

Property **(A1)** is satisfied since the function $\widehat{k}(s)$, defined in (42) admits a meromorphic extension to Σ_π (the whole complex plane cut along the negative real axis) and $\theta_0 < \pi/2$. Further, since A generates a bounded analytic semigroup of angle $\pi/2$ it follows that $\Sigma_\pi \subseteq \rho(A)$ and for any $\phi < \pi$ there exists a constant $C = C(\phi)$, such that

$$\|(s - A)^{-1}\| \le C/|s| \quad \text{for all} \quad s \in \Sigma_\phi. \tag{49}$$

On the other hand, the function $g(s)$ defined by (40) obeys the inequality (see [10], Lemma 3.1)

$$|\arg g(s)| \le (1+\alpha)|\arg s|. \tag{50}$$

Therefore, if $s \in \Sigma_{\theta_0+\pi/2}$ then, by (46), $|\arg g(s)| \le (1+\alpha)(\theta_0 + \pi/2) < \pi$, *i.e.*, $g(s) \in \Sigma_\pi \subseteq \rho(A)$. This implies **(A2)**. To prove **(A3)**, let $s \in \Sigma_{\theta+\pi/2}$ for some $\theta < \theta_0$. Then, (50) implies $g(s) \in \Sigma_\phi$, where $\phi = (1+\alpha)(\theta + \pi/2) < \pi$. Applying (49) we deduce

$$\|(g(s) - A)^{-1}\| \le C/|g(s)| \quad \text{for all} \quad s \in \Sigma_{\theta+\pi/2},$$

where the constant C depends on θ via the relation between ϕ and θ. This together with the definition (47) of the function $H(s)$ implies (48) and thus **(A3)** is also established.

In this way we proved that problem (38) admits an analytic resolvent. This, in particular, implies that equation (38) is a parabolic Volterra integral equation, see [15], Chapter 3.

After these very general results, let us consider the simplest version of equation (37).

THE SCALAR EQUATION AND THE BEHAVIOUR OF ITS SOLUTION

In this section we study the scalar case of problem (37), where the operator A is replaced by a constant $-\lambda$, $\lambda > 0$:

$$(1 + aD_t^{\alpha})u_t = -\lambda(1 + bD_t^{\beta})u,\ t > 0, \quad u(0) = 1,\ (J_t^{1-\alpha}u_t)(0) = 0, \tag{51}$$

where $a > 0$, $b \geq 0$, $\alpha, \beta \in (0,1)$. For $\lambda = \mu\lambda_n$ the solution of problem (51) coincides with the time-dependent component $G_n(t)$ in the eigenfunction expansion (27). Therefore, the behaviour of the solution of (51) determine the character of time evolution (decay in time, oscillations, etc.), as well as the smoothing effect in space for problem (23)-(24)-(25)-(36). In [10] some estimates of the solution of (51) are obtained, which can serve as a basis for establishing of regularity in space. Here we concentrate our attention on the behaviour of the solution in time.

Note that in the case $a = 0, b > 0$, the scalar equation (51) as well as the abstract equation (37) are already well studied, see [11] and [12]. It is proven that the solution of the scalar equation is nonnegative and nonincreasing for any $t \geq 0$, moreover, it is a completely monotone function. In this case $g(s)$ is a Bernstein function, which implies that the solution of the abstract problem (37) is positivity preserving, *i.e.,* $v \geq 0$ always implies $u(t) \geq 0$ for all $t \geq 0$.

By applying the Laplace transform we obtain in the same way as when deriving (34) the following identity for the Laplace transform of the solution u of (51):

$$\widehat{u}(s) = \frac{1 + as^{\alpha}}{s(1 + as^{\alpha}) + \lambda(1 + bs^{\beta})}.$$

Based on this identity we first derive a series representation of $\widehat{u}(s)$ for sufficiently large $|s|$. Then, taking the inverse Laplace transform and using (10), we obtain the following series representation for $u(t)$:

$$u(t) = \sum_{k=0}^{\infty}\sum_{p=0}^{k}\sum_{j=0}^{p}(-1)^k\binom{k}{p}\binom{p}{j}a^{-(k+1)}\lambda^{k-p-j}b^j\left(\omega_{(\alpha+1)(k+1)-p+(1-\beta)j}(t) + a\omega_{(\alpha+1)k-p+(1-\beta)j+1}(t)\right). \tag{52}$$

(For more details on using this technique for obtaining of series representations see, *e.g,* [20].) Representation (52) is used here for the numerical computation and visualization of the solution $u(t)$ of problem (51). It should be noted that this representation is not appropriate for large times, *i.e.,* this method imposes restrictions on the time interval, in which the numerical computation of $u(t)$ is possible, as those restrictions appear to be more severe for $\alpha < \beta$. This can be seen on Figures 2 and 3, where plots of the solution $u(t)$ are presented for different values of the parameters. The time intervals are chosen to be the maximal possible before reaching a time, where the computations based on formula (52) become unstable. We see that the time intervals for $\alpha < \beta$ (Figure 3) are much shorter than for $\alpha > \beta$ (Figure 2, right).

It is seen on Figure 2 that for $b = 0$ the solutions $u(t)$ exhibit damped oscillations, as the oscillations are more pronounced for larger α, $\alpha \to 1$. When $b > 0$ and $\alpha > \beta$ similar behavior is observed, but with less pronounced oscillations.

For $\alpha < \beta$ the plots on Figure 3 show a different behavior. Let us notice first that the plots confirm the theoretical observation $u'(0) = \infty$ for $\alpha < \beta$, which was the reason for changing the second initial condition from (26) to (31). Further, the plots show a positive, monotonically decreasing function for $\alpha < \beta$ (at least in the time region, in which computation based on (52) was possible). The question is whether this is the real behavior for all $t > 0$ in this case.

To get more information about the solution behavior for $\alpha < \beta$ we refer to a result proven in [15], Proposition 4.5, which implies the following relation between the solution $u(t)$ of (51) and the corresponding function $g(s)$ defined by (40): The function $u(t)$ is positive and nonincreasing with respect to $t > 0$ for every $\lambda > 0$ if and only if $g(s) \in \mathcal{BF}$.

This statement is very useful, since we can derive some properties of the solution $u(t)$ from properties of the function $g(s)$, which admits a much simpler expression. Let us note first that if $a = 0$ then $g(s) \in \mathcal{BF}$ and $u(t) \in \mathcal{CMF}$, which is in agreement with the above assertion.

To check whether $g(s)$ can be a Bernstein function if $a > 0$, we note first that

$$g(s) \sim ab^{-1}s^{\gamma},\ \text{as } s \to \infty,$$

where $\gamma = 1 + \alpha - \beta$. Since $\gamma \in (0,1)$ for $\alpha < \beta$, then for sufficiently large $s > 0$ the function $g(s)$ behaves as a Bernstein function if $\alpha < \beta$ and $g(s) \notin \mathcal{BF}$ if $\alpha > \beta$. The last observation is also seen from the plots of $g(s)$ on Figure 4 for $\alpha > \beta$, since any Bernstein function should be nonnegative, monotonically nondecreasing ($g' \geq 0$) and

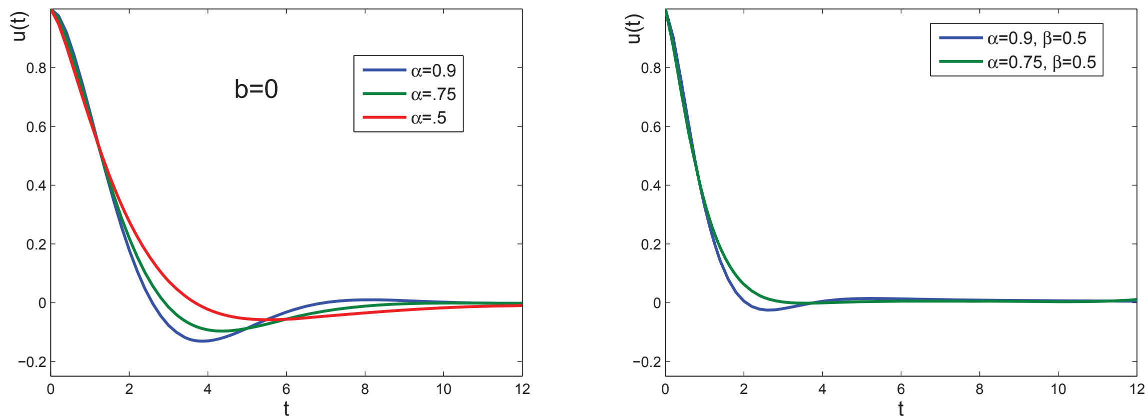


FIGURE 2. Plots of the solution $u(t)$ for $\lambda = 1$; $a = 1$, $b = 0$ and different values of α (left); $a = b = 1$ and $\alpha > \beta$ (right)

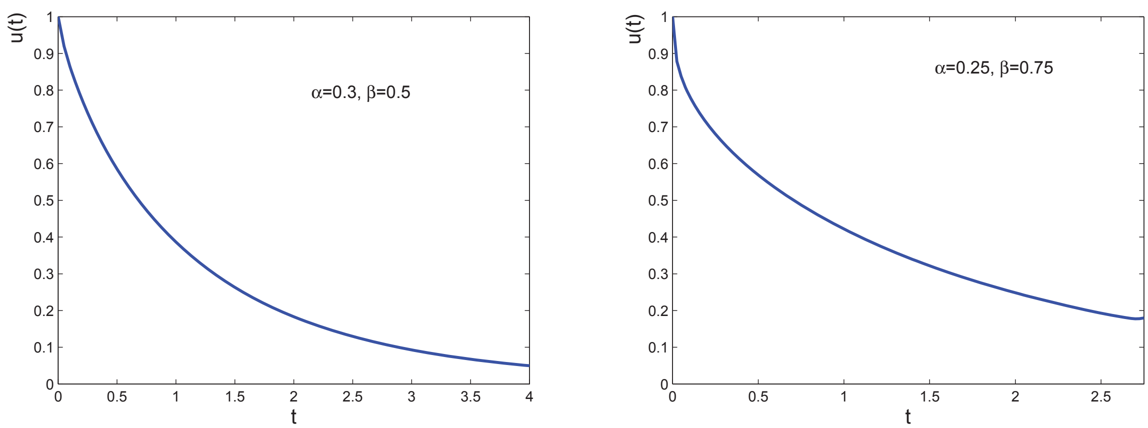


FIGURE 3. Plots of the solution $u(t)$ for $\lambda = 1$, $a = b = 1$ and $\alpha < \beta$.

concave ($g'' \leq 0$). Note that the observed oscillations in $u(t)$ on Figure 2 when $\alpha > \beta$ are also in agreement with the fact that $g(s) \notin \mathcal{BF}$ for $\alpha > \beta$.

Regarding the case $\alpha < \beta$ if we examine the function $g'(s)$, we see on Figure 5 that it is increasing for small s (note that $g'(0) = 1$). Indeed, from the expression (40) for $g(s)$

$$g''(s) \sim a(\alpha + 1)\alpha s^{\alpha - 1}, \text{ as } s \to 0, \quad \alpha < \beta.$$

Therefore $g''(s) > 0$ as $s \to 0$ and thus $g(s) \notin \mathcal{BF}$ also when $\alpha < \beta$.

Therefore, according to the relationship between the properties of $g(s)$ and $u(t)$ stated above, the solution $u(t)$ of the scalar equation (51) can not be positive and nonincreasing for all $t > 0$, *i.e.,* there exists some $t_0 > 0$ such that $u(t_0) \leq 0$ or $u'(t_0) > 0$.

Obviously, the time interval considered on Figure 3 for $\alpha < \beta$ does not contain such points t_0 and another numerical algorithm for the computation of the solution of problem (51) is needed, which works for sufficiently large times.

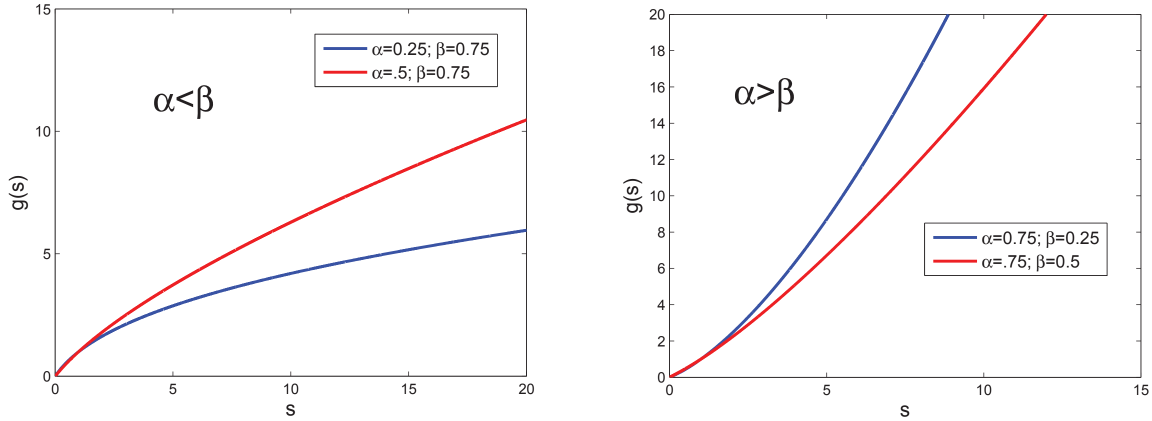


FIGURE 4. Plots of the function $g(s)$ for $a = b = 1$; $\alpha < \beta$ (left) and $\alpha > \beta$ (right)

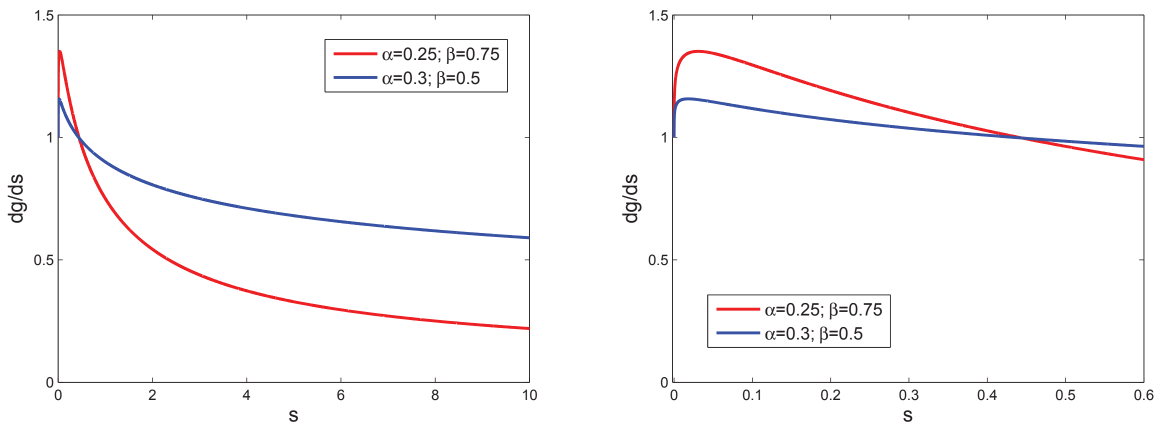


FIGURE 5. Plots of the function $g'(s)$ for $a = b = 1$ and $\alpha < \beta$

CONCLUSIONS

The time evolution of the velocity field of a unidirectional fractional Oldroyd-B fluid studied in this work exhibits a character intermediate between the two limiting cases: fractional second grade fluid ($a = 0$, positive solutions, monotonically decreasing in time) and fractional Maxwell fluid ($b = 0$, damped oscillations). For $1 > \alpha > \beta > 0$ the behaviour is similar to those in the case $b = 0$: still some damped oscillations are observed, which are more pronounced for larger α, although in general, the presence of the term bD_t^{β} enhances the damping effect. For $0 < \alpha < \beta < 1$ the presented plots for small times show behaviour similar to those in the case $a = 0$. However, theoretical analysis shows that this can not hold for all $t > 0$. Therefore, it is desirable to construct a numerical algorithm for the computation of the solutions in this case, which works for sufficiently large times and is able to capture the peculiarities in the solution behaviour for $\alpha < \beta$ predicted by the theoretical analysis.

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